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Asymptotics of homogeneous oscillations in a compressible viscous fluid

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— Dedicated to Constantine Dafermos on his 60^{th} birthday.

Abstract. The large amplitude high frequency oscillations of the flow of a compressible viscous fluid have been shown to obey to an integro-differential system. We consider here the case of homogeneous oscillations, with both gas-like pressure law and a van der Waals one. We show that the solution admits a limit as the time increases. This limit is constant in the former case, but may take up to three distinct values in the latter.

Keywords: Van der Waals gas, Lasalle's invariance principle.

Mathematical subject classification: 34D05, 76N15, 76T10.

The flow of a compressible viscous fluid is described by its velocity u and specific volume v. In one space variable, we may use Lagrangian coordinates (x, t), so that the flow obeys the Navier-Stokes equations

$$v_t = u_x, \tag{1}$$

$$u_t + p(v)_x = (\mu(v)u_x)_x. \tag{2}$$

Hereabove, p is the pressure and μ is a relative viscosity ; these are given positive smooth functions of v.

The Cauchy problem on the whole line $\mathbb{R}$ or in bounded domains with various boundary conditions is well understood under natural assumptions. See the monograph by P.-L. Lions [7] for a broad bibliography. A major feature is that the density is not smoothed out by this partly parabolic problem, while u and

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the stress $\mu u_x - p$ are, in the sense that their x-derivatives are L^2, locally in space-time variables, even though the initial data may only be bounded. This fact reflects in the propagation of discontinuities along the particle paths x =cst, first observed by D. Hoff & J. Smoller [5]. Similarly, high-frequency oscillations present in the initial data do propagate, even when their amplitude is not small.

This propagation was studied rigorously in [9] (see also an asymptotic analysis by russian authors [1, 2]). Let (v^ϵ, u^ϵ) be a sequence of solutions, associated to data $(v_0^\epsilon, u_0^\epsilon)$ which converge in the sense of Young (therefore in L^∞ weak-star). Denoting by u the strong limit of u^ϵ, by σ the weak-star limit of the stress and by $V(x, t, y)$ the Young limit of v^ϵ, defined by

$$\lim_{w*} f(v^\epsilon) = \int_0^1 f(V(\cdot, \cdot, y))dy, \quad \forall f \in C,$$

(a non-decreasing V is uniquely defined), the oscillations obey to the evolution system

$$V_t = \frac{\sigma + p(V)}{\mu(V)}, \tag{3}$$

$$u_t = \sigma_x, \tag{4}$$

$$u_x = \sigma \int_0^1 \frac{dy}{\mu(V)} + \int_0^1 \frac{p(V)dy}{\mu(V)}. \tag{5}$$

Let us point out that only the first line involve all the variables (x, t, y). Also notice that the third line serves only as a definition of σ in terms of the primary variables (V, u).

We focus in this paper on the homogeneous oscillations, that is on the solutions of (3,4,5) such that u, V are independent on the space variable x. This occurs when the data do not depend on x. We easily see that u must be a constant and the system reduces to an integro-differential equation

$$\mu(V)\left(\int_0^1 \frac{dy}{\mu(V)}\right)\partial_t V = p(V)\int_0^1 \frac{dy}{\mu(V)} - \int_0^1 \frac{p(V)dy}{\mu(V)}, \quad y \in (0, 1). \tag{6}$$

It was already pointed by D. Hoff (see [3, 4]) that for a gas-like medium (that is $p' < 0$), the amplitude of discontinuities in v decays, though it might increase for a van der Waals fluid, for which the sign of p' depends on the state v. We wish to clarify this point at the level of oscillations. More specifically, we raise the question of the asymptotic behaviour of the solutions of (6), given the initial data. We shall prove, under a non-degeneracy assumption, that $V(\cdot, t)$ admits a unique limit $\hat{V}$.

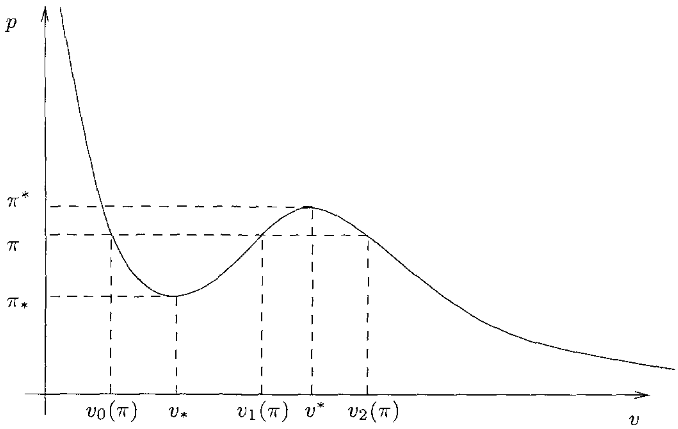

Figure 1: The typical equation of state for a van der Waals fluid.

We first describe the rest points for the system (6), since $\hat{V}$ must be such a state. It is clear that $y \mapsto p(\hat{V})$ must be a constant, that we denote by π. We consider two types of fluid. The first one is a gas, meaning that $p' < 0$ everywhere, with $p(0) = +\infty$ and $p(+\infty) = 0$. The second one is a van der Waals fluid, for which still $p(0) = +\infty$ and $p(+\infty) = 0$, but $p' < 0$ on $(0, v_*)$ and $(v^*, +\infty)$, while $p' > 0$ on (v_*, v^*). The intervals $(0, v_*)$, $(v^*, +\infty)$, (v_*, v^*) correspond to liquid, gas and spinodal region respectively. For a gas, $\hat{V}$ is simply a constant, while it may take up to three distinct values for a van der Waals fluid. Denoting by v_0, v_2, v_1 the inverse functions of p on each of the intervals $(0, v_*)$, $(v^*, +\infty)$, (v_*, v^*), $\hat{V}$ will have the form

$$\hat{V}(y) = \begin{cases} v_0(\pi), & y \in (0, \alpha_0), \\ v_1(\pi), & y \in (\alpha_0, \alpha_0 + \alpha_1), \\ v_2(\pi), & y \in (\alpha_0 + \alpha_1, 1), \end{cases} \tag{7}$$

since it is non-decreasing. Denoting by $\alpha_2 := 1 - \alpha_0 - \alpha_1$, we have

$$\int_0^1 f(\hat{V})dy = \sum_{j=0}^{2} \alpha_j f \circ v_j(\pi), \quad \forall f \in C.$$

In the sequel, we use the notation $\pi_* = p(v_*)$, $\pi^* = p(v^*)$.

Let $V_0 \in L^\infty(0, 1)$ be an initial data, such that $\inf V_0 > 0$. We consider the Cauchy problem for (6). A local solution exists and is unique, thanks to the

Cauchy-Lipschitz theory of ODEs. Moreover, for both types of fluid, we may choose an interval $[v_-, v_+]$, such that

$$0 < v_- < \inf_y V_0(y), \quad \sup_y V_0(y) < v_+ < +\infty$$

and

$$p(v_-) < p(v) < p(v_+), \quad \forall v \in (v_-, v_+).$$

Then the domain X, formed of measurable function W, defined on $(0, 1)$, with values in (v_-, v_+), is positively invariant and (6) defines a continuous semi-group on X, when endowed with the topology of $L^p(0, 1)$ $(1 \leq p \leq \infty)$. Therefore, the solution is globally defined for $t > 0$. We are interested in the asymptotics of V as $t \to +\infty$.

Our main result reads as follows.

Theorem 1. *Let $V_0 \in L^\infty(0, 1)$ be a non-decreasing initial data with $\inf_y V_0(y) > 0$. Let $V(y, t)$ be the corresponding global solution of the Cauchy problem for (6). The fluid may be either a gas or a van der Waals fluid; in the latter case, we assume that*

> **(H)**: *the only intervals $[p_-, p_+]$ $(\subset [\pi_*, \pi^*])$, on which v_0, v_1, v_2 and the constants are linearly dependent functions, are trivial $(p_- = p_+)$.*

Then $(V(\cdot, t))_{t\to+\infty}$ admits a unique limit in the L^p-norm, for $1 \leq p < \infty$.

Remarks.

- The same question was already considered by Pego in [8]. He was able to remove the assumption **(H)**. However, he was only able to treat the case where V_0 is a step function. Equivalently, $\{(0, 1), dx\}$ was replaced by a finite set with an atomic measure. Our proof below works actually for every probability space.

- Thanks to the hypothesis **(H)**, our proof is more elementary than Pego's, using only Lasalle's invariance principle and explicit calculations.

The rest of the note is devoted to the proof.

We first remark that $V(\cdot, t)$ is non-decreasing in y for all time, so that $V(\cdot, t)$ has a total variation bounded by $v_+ - v_-$. This implies the relative compactness of $(V(t))_{t\geq 0}$ in L^p. Therefore it will be sufficient to prove that its omega-limit set Ω is a singleton. For that, we shall use Lasalle's invariance principle [6]. This tells first that if L is a C^1 Liapunov function of the flow, that is

$$\frac{d}{dt}L[V] = R[V] \leq 0, \tag{8}$$

then Ω is contained in a level set of L : there exists a number ν_L such that $L[W] = \nu_L$ for all W in Ω. Next, it tells that the decay rate R is zero on Ω. Last, Ω must be a connected set for the L^p-topology, since the flow is continous with respect to it. We apply Lasalle's principle to functionals of the form

$$L_F[V] := \int_0^1 F(V(y))dy,$$

where $F' = f \circ p$. Such functionals satisfy (8) whenever f is non-increasing. The decay rate is then given by

$$\left(\int_0^1 \frac{dy}{\mu(V)}\right) R[V] = \int_0^1 \frac{f \circ p(V)p(V)dy}{\mu(V)} \int_0^1 \frac{dy}{\mu(V)} - \int_0^1 \frac{f \circ p(V)dy}{\mu(V)} \int_0^1 \frac{p(V)dy}{\mu(V)}.$$

Applying this with $f(s) = -s$ (for which F is the internal energy of the fluid), and using the Cauchy-Schwarz inequality, we see that Ω is made of rest states only.

We immediately conclude in the case of a gas. Actually, the conservation law

$$\frac{d}{dt}\int_0^1 V dy = 0$$

gives the additional information that

$$\int_0^1 W(y)dy = \int_0^1 V_0(y)dy =: \ell, \quad \forall W \in \Omega. \tag{9}$$

Since here any W of Ω must be constant, we see that $W \equiv \ell$, so that Ω is a singleton.

We may now restrict to the case of a van der Waals fluid. From Lasalle's principle, there exists a number ν_F for each F of the form $F' = f \circ p$ with f non-increasing, such that

$$L_F[W] = \nu_F, \quad \forall W \in \Omega. \tag{10}$$

Since every function of bounded variation is the difference of two non-increasing functions, this is still true when $f \in BV(v_-, v_+)$.

We split our analysis into three cases. Recalling that elements W of Ω are rest states, for which $p \circ W$ is some constant π, we first assume that Ω contains such a state with $\pi > \pi^*$ (or as well $\pi < \pi_*$). Then W is a constant and it is an isolated rest point in the hyperplane defined by the constraint (9). Since Ω is connected, it must be equal to the singleton $\{W\}$ and we are gone.

The second case is the one where an element W of Ω is such that $\pi_* < \pi < \pi^*$. Then nearby points $\bar{W}$ in Ω must be of the same form, with a pressure $\bar{\pi} \in (\pi_*, \pi^*)$ and lengths β_j instead of α_j. Let denote by v_j and $\bar{v}_j$ the points $v_j(\pi)$ and $v_j(\bar{\pi})$ in the following calculations. Using the lemma 1 below, we see that the triplet $\vec{\alpha}$ is uniquely determined in terms of π and ℓ, using the equations

$$\sum_j \alpha_j = 1, \quad \sum_j \alpha_j v_j(\pi) = \ell, \quad \sum_j \alpha_j F(v_j(\pi)) = \nu_F.$$

Now, if Ω is not a singleton, we may choose $\bar{W}$ as above, and we conclude that $\bar{\pi}$ must be distinct from π. We always may assume that $\pi < \bar{\pi}$. Then we have $\bar{v}_0 < v_0 < v_1 < \bar{v}_1 < \bar{v}_2 < v_2$. From (10), we write $L_F[W] = L_F[\bar{W}]$:

$$\sum_j \alpha_j F(v_j) = \sum_j \beta_j F(\bar{v}_j).$$

in other words,

$$\left(\alpha_2 \int_{\bar{v}_2}^{v_2} +(\alpha_2 + \beta_0 - 1) \int_{v_1}^{\bar{v}_1} +\beta_0 \int_{\bar{v}_0}^{v_0} +(\alpha_2 - \beta_2) \int_{\bar{v}_1}^{\bar{v}_2} +(\beta_0 - \alpha_0) \int_{v_0}^{v_1}\right) f \circ p(v)dv = 0.$$

Since this holds for all f in BV, we deduce that $\alpha_2 = \beta_2, \alpha_0 = \beta_0$ (that is $\vec{\alpha} = \vec{\beta}$) and also

$$\left(\alpha_2 \int_{\bar{v}_2}^{v_2} -\alpha_1 \int_{v_1}^{\bar{v}_1} +\beta_0 \int_{\bar{v}_0}^{v_0}\right) f \circ p(v)dv = 0.$$

This last equality amounts to

$$\int_{\pi}^{\bar{\pi}} \sum_j \frac{\alpha_j}{p' \circ v_j} f(p)dp = 0.$$

Since this holds for all f in BV, we deduce that

$$\sum_j \frac{\alpha_j}{p' \circ v_j} \equiv 0 \quad \text{on} \quad (\pi, \bar{\pi}),$$

which contradicts **(H)**. Therefore, Ω is a singleton.

In our last case, the pressure of elements of Ω can only take the values π_* or π^*. Since Ω is connected, it takes only one of these values and this shows as above that W is unique. □

We complete the proof of the theorem with the

Lemma 1. *Let f be strictly decreasing (or strictly increasing). Let choose $\pi \in (\pi_*, \pi^*)$. Then*

$$\det \begin{pmatrix} 1 & 1 & 1 \\ v_0(\pi) & v_1(\pi) & v_2(\pi) \\ F(v_0(\pi)) & F(v_1(\pi)) & F(v_2(\pi)) \end{pmatrix} \neq 0.$$

The vanishing of the determinant would mean that the three points $(v_j, F(v_j))$ are aligned. Since v_1 lies between v_0 and v_2, we write $v_1 = (1-\theta)v_0 + \theta v_2$ with $\theta \in (0, 1)$. We must prove that $F(v_1) - (1-\theta)F(v_0) - \theta F(v_2)$ is non-zero. However, this equals

$$(1-\theta)\int_{v_0}^{v_1} f \circ p(v)dv - \theta \int_{v_1}^{v_2} f \circ p(v)dv.$$

Since $p < \pi$ on (v_0, v_1) and $p > \pi$ on (v_1, v_2), and f is decreasing, this quantity is strictly positive. □

Remarks.

- For a van der Waals gas, **(H)** is generically satisfied. When it is not, the numbers m such that there exists a linear combination $\sum_j \gamma_j v_j(\pi) \equiv m$, with $\gamma_j \geq 0$ and $\sum_j \gamma_j = 1$, form an at most denumerable set M. When the data V_0 is such that $\ell \notin M$, the convergence to a single equilibrium state still holds.

- A typical example of a pressure law which violates **(H)** is a cubic polynomial (on some restricted interval). For instance, one may consider $p(v) := -v^3 + 4v^2 - 5v$, with $\mu(v) = 1/v$ (constant viscosity). Here, we have $v_0 + v_1 + v_2 \equiv 4$. Though the previous analysis does not end to a conclusion when $\ell = 4/3$, we have been able to prove directly the convergence, at least when the initial data consists in three constant states on intervals of lenghts $1/3$. This is a particular case of Pego's result [8]. However, the general question of convergence when **(H)** fails is still open.

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